Bond Rating Enhancement Using Corporate Bond Default Data: A New Methodology

Financial Analysts Journal, August 2000

Sankarshan Acharya *

Mini Abstract: This paper proposes a new approach, based on corporate bond default data, to determine the quantity of new equity capital that a bond issuer must raise or the amount of current debt it must retire to consistently enhance its bond ratings. The current rating industry approach to determine such standards for upgrading bond ratings assumes that the probability of bond default by the issuer is unaffected even after new equity capital is raised or debt is retired. Our specification relaxes this assumption. Our estimates show dramatically different bond rating upgrade standards from those implied by the industry approach.

*(Phone: 312 413 9204, e-mail: sacharya@uic.edu), University of Illinois at Chicago, 601 South Morgan Street, Chicago, IL 60607. I am very thankful to John Mingo (Federal Reserve Board) for his comments on a previous draft of the paper and, especially, to Jeromy Fons (Moody's Investors Service) for providing the data and for generous discussions, and to an anonymous referee for comments.

Abstract: This paper proposes a new approach, based on corporate bond default data, to determine the quantity of new equity capital that a bond issuer must raise or the amount of current debt it must retire to consistently enhance its bond ratings. The current rating industry approach to determine such standards for upgrading bond ratings assumes that the probability of bond default by the issuer is unaffected even after new equity capital is raised or debt is retired. Our specification relaxes this assumption. Our estimates show dramatically different bond rating upgrade standards from those implied by the industry approach.

Author Digest: Since losses on bonds depend on the future asset value, we specify a distribution for the random future value of assets. Consistent with a large body of finance literature and practical applications, we assume that the future payoff (return per dollar of current asset value) to assets backing bonds in each rating category is distributed lognormally. Thus, the random (continuously compounded) rate of return on investment in assets of an entity has a normal distribution. Different return distributions for assets backing differently rated bonds make default probabilities and expected bond loss rates specific to each rating category.

In the *first step* of our procedure, we estimate the parameters of the lognormal distributions using the observed means and variances of losses on bonds in each rating category obtained from Moody's corporate bond default data. The theoretical mean and variance of loss are well defined functions (derived in the next section) of parameters of the lognormal asset value distribution. We equate these theoretical functions, respectively, to the estimated mean and variance of losses to solve for the two parameters of the asset value distribution for each rating category. The parameter estimates identify the underlying asset value distribution for each rating category. In our *second step*, we use these identified distributions to solve for the increase in the new equity capital infusion needed to reduce the current expected loss on bonds of an issuer to that for higher rated bonds. This is possible because the expected loss on bonds is a decreasing function of the current value of assets and hence of the new equity capital infused by a bond issuer.

We calculate new equity capital infusions for enhancing ratings of bonds aged three, four and five years where ages are reckoned from the date of issuance. We demonstrate the method of estimating new equity capital infusions for enhancing ratings B, Ba, Baa, A, or Aa to Aaa. Our new capital infusion estimates are much smaller than those obtained by the standard approach. For example, to enhance bond rating Baa to Aaa, the issuer must raise new equity capital equal to 3.02 percent of its current assets within our approach, as opposed to 55.71 percent under the standard approach (reported in Tables 3 and 4 for bonds aged 3 years). This is mainly because our approach accounts for the fact that the default probability decreases due to new equity capital infusion.

Our methodology can help the rating industry revise its bond rating standards. It can also be applied to set bank capital standards if bank assets can be rated. Bank regulators have recently expressed a serious interest to enforce such standards by requiring banks to have their assets rated by public rating agencies. The idea behind this interest is that rating of bank assets by reputed public rating agencies can more credibly reflect asset quality than the current bank examination ratings, and capital standards can be reliably based on more accurate asset quality estimates. Regulators have recently proposed to enhance bank asset quality to an equivalent of Aa rating. Depending on the existing quality of a bank's assets and the rating by a public rating agency (that will be known after the system is implemented), one can use our procedure to estimate the new equity capital that the bank must raise to enhance its rating to Aa.

1 Introduction

The rating industry rates bonds based on the value and risk of assets that are legally available as collaterals for the bonds.[1] Firms have traditionally issued bonds as claims against all of their assets. Many firms now form legally independent entities (such as "trusts" and "conduits") to issue bonds backed by an entity's assets.[2] The rating industry often calls such entities bankruptcy remote because their owner (a firm) does not necessarily become bankrupt even when an entity does. At the time of bankruptcy of a legally independent entity, assets of this entity only are used to cover its debt obligations. The owner is not legally obligated to use assets from other entities to pay debtholders of a bankrupt entity. To upgrade or downgrade bond ratings of an issuer, rating agencies consider marginal changes in the value and risk of assets of the issuer. This paper proposes a new approach for consistently rating bonds and for upgrading or downgrading ratings based on the value and risk of assets backing the bonds.

A main purpose of the rating industry as well as of this paper is to calculate the new equity capital, denoted C_l^h, for upgrading l-rated bonds to h-rated.[3] To convince that a new approach is needed in practice, we first describe the standard approach followed by the bond rating industry.[4] The standard approach uses two pieces of information derived from corporate bond default data: (a) the severity of loss defined as the expected loss per dollar owed to bondholders, given that a loss will occur, denoted x; and (b) the probability of a loss over a future time period, given a current rating R, denoted $p(R)$. Moody's Special Report [MSR] (1993, page 8) on corporate bond defaults documents a severity of loss estimate $x = .6$, using a proprietary comprehensive corporate bond database over the period 1972-1992. MSR also reports the probability of a loss (rates of default) on bonds in each rating category. The expected future loss on bonds rated R is $x\,p(R)$. For an illustration of the rating industry's approach, suppose that a legally independent entity (bond issuer) wants to raise new capital to upgrade the existing bond rating (say B) to a higher one (say Aa). This approach assumes that the expected future loss to bondholders depends only on the ratings. It calculates C_B^{Aa} by equating the expected future loss per dollar of bonds of a B-rated issuer after it raises new capital C_B^{Aa} to that of Aa-rated

[1] By the bond rating industry, we mean large rating agencies (e.g., Moody's, S&P and Fitch) whose policies for rating are publicly available in printed brochures available from the agencies.

[2] Firms generally report the assets held in bankruptcy remote entities as off-balance sheet items. For example, Citicorp had about $40 billion (about 20 percent of its assets) in conduits, trusts and master trusts as of 1994.

[3] Without loss of generality, we can compute an equivalent amount of debt reduction to attain the same rating enhancement.

[4] By the standard approach, we mean the reasonably well documented procedure, e.g., in Moody's Structured Finance Special Report (1994). To the best of the author's knowledge, neither the academic literature nor the industry literature has suggested the approach proposed here for determining the amount of new equity capital infusion necessary to improve the bond rating of an issuer.

bonds. If the B-rated bond issuer increases its equity capital (and hence asset value) by C_B^{Aa} dollars, its severity of loss will be reduced from x to $x - C_B^{Aa}$ and its expected loss on bonds will be decreased from $x\,p(B)$ to $(x - C_B^{Aa})p(B)$. Since the expected loss on Aa-rated bonds is equal to $x\,p(Aa)$, the rating industry determines C_B^{Aa} by solving $(x - C_B^{Aa})p(B) = x\,p(Aa)$, i.e., $C_B^{Aa} = x(1 - p(Aa)/p(B))$.[5] These solutions are later reported using raw data on probability of losses for differently rated bonds.

While the rating industry's approach is simple, it makes a strong assumption that the probability of a loss on bonds stays constant even after the asset value backing the bonds is increased via new equity capital infusion. For example, the industry approach implies an asset value increase of 55.71 percent to upgrade Baa-rated bonds to Aa-rated (as reported in Table 4 for bonds aged 3 years reckoned from the date of issuance). This calculation assumes that the probability of a loss on Baa-rated bonds does not change even after the bond issuer increases its asset value by as much as 55.71 percent over a one-year horizon. This unrealistic assumption seems to result in dramatically large amounts of new equity capital estimates for rating enhancement, as compared to the estimates under the new approach, especially, for better rated bonds (Ba and above).

Both the industry's approach and the one proposed here quantify credit quality (bond rating) by the expected loss on bonds and hence calculate C_l^h by equating the expected future loss per dollar of bonds of an l-rated issuer raising new capital C_l^h to that of h-rated bonds. There are three principal differences, however, between the procedure proposed here and the standard one for calculating expected loss; the first two differences are conceptual and the third is methodological. First, our procedure accounts for changes in the distribution of losses on bonds of an issuer due to new equity capital infusion. Second, our approach is based on an economic rationale for the timing of corporate bond defaults, namely, bondholders incur a loss only when the issuer's asset value falls below the amount owed. This means that the default event is generated endogenously within our model, contrasting the assumption underlying the standard approach that this event occurs whenever the realization of an exogenous latent random variable falls below an exogenous threshold.[6]

Since losses on bonds depend on the future asset value, we need to specify a distribution for the random future value of assets. Consistent with a large body of finance literature and practical applications, we assume that the future payoff (return per dollar of current asset

[5] See, e.g., "Appendix A: Mathematics of Calculating Portfolio Credit Risk" in Moody's Investors Service Structured Finance Research and Company, 1991, March, New York, NY 10007.

[6] The economic rationale for an endogenous default event circumvents the Lucas (1976) critique of *ad hoc* specifications.

value) to assets backing bonds in each rating category is distributed lognormally. Thus, the random (continuously compounded) rate of return on investment in assets of an entity has a normal distribution. Different return distributions for assets backing differently rated bonds make default probabilities and expected bond loss rates specific to each rating category.

In the *first step* of our procedure, we estimate the parameters of the lognormal distributions using the observed means and variances of losses on bonds in each rating category obtained from Moody's corporate bond default data. The theoretical mean and variance of loss are well defined functions (derived in the next section) of parameters of the lognormal asset value distribution. We equate these theoretical functions, respectively, to the estimated mean and variance of losses to solve for the two parameters of the asset value distribution for each rating category. The parameter estimates identify the underlying asset value distribution for each rating category. In our *second step*, we use these identified distributions to solve for the increase in the new equity capital infusion needed to reduce the current expected loss on bonds of an issuer to that for higher rated bonds. This is possible because the expected loss on bonds is a decreasing function of the current value of assets and hence of the new equity capital infused by a bond issuer.

We calculate new equity capital infusions for enhancing ratings of bonds aged three, four and five years where ages are reckoned from the date of issuance. We demonstrate the method of estimating new equity capital infusions for enhancing ratings *B*, *Ba*, *Baa*, *A*, or *Aa* to *Aaa*. Our new capital infusion estimates are much smaller than those obtained by the standard approach. For example, to enhance bond rating *Baa* to *Aaa*, the issuer must raise new equity capital equal to 3.02 percent of its current assets within our approach, as opposed to 55.71 percent under the standard approach (reported in Tables 3 and 4 for bonds aged 3 years). This is mainly because our approach accounts for the fact that the default probability decreases due to new equity capital infusion.

Our methodology can help the rating industry revise its bond rating standards. It can also be applied to set bank capital standards if bank assets can be rated. Bank regulators have recently expressed a serious interest to enforce such standards by requiring banks to have their assets rated by public rating agencies. The idea behind this interest is that rating of bank assets by reputed public rating agencies can more credibly reflect asset quality than the current bank examination ratings, and capital standards can be reliably based on more accurate asset quality estimates. Regulators have recently proposed to enhance bank asset quality to an equivalent of *Aa* rating. Depending on the existing quality of a bank's assets and the rating by a public rating agency (that will be known after the system is implemented), one can use our procedure to estimate the new equity capital that the bank must raise to enhance its rating to *Aa*.

2 The Methodology

In this section, we describe a two-step procedure to determine the amount of new equity capital that a bond issuer (legally independent entity) must raise to enhance its bond rating. This procedure uses a rich source of corporate bond default data such as that used in Moody's Special Report.

2.1 Mean and Variance of Underlying Asset Return

Consider an R-rated bond issuer over a one-period horizon with F dollars of end-of-period bond obligation (face value of bonds) and a unit of beginning-of-period investment in assets. The random end-of-period asset value, denoted $v(R)$, is the return (payoff) per unit investment in assets. Given that the issuer defaults on the bonds, the random end-of-period loss will be $F - v(R)$ on all bonds and $1 - V(R)$ per bond for $V(R) \equiv v(R)/F$. The loss on bonds is obviously zero if the issuer does not default at the end of the period. Thus, the unconditional mean bond loss rate (loss per bond) can be expressed as:

$$E[\max\{0, 1 - V(R)\}] \equiv \mu(R), \tag{1}$$

where the future asset value per bond, $V(R)$, is assumed to be lognormally distributed, i.e., $ln(V(R))$ is normally distributed with mean and standard deviation, denoted $\tilde{\mu}(R)$ and $\tilde{\sigma}(R)$, respectively. The unconditional variance of bond loss rate is given by

$$Var[\max\{0, 1 - V(R)\}] \equiv \sigma(R)^2. \tag{2}$$

We demonstrate that the lognormal distribution parameters $(\tilde{\mu}(R),\ \tilde{\sigma}(R))$ can be statistically identified from observed unconditional mean and variance of bond loss rates for each rating category R available from Moody's Special Report. The first step of our methodology is to estimate these identified parameters $(\tilde{\mu}(R),\ \tilde{\sigma}(R))$.

We can derive formulas for the unconditional mean and variance of bond loss rate by evaluating (1) and (2) for a lognormal distribution as follows (see Appendix):

$$\mu(R) = \Phi\left(-\frac{\tilde{\mu}(R)}{\tilde{\sigma}(R)}\right) - Exp\left(\tilde{\mu}(R) + \frac{\tilde{\sigma}(R)^2}{2}\right)\Phi\left(-\frac{\tilde{\mu}(R) + \tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right), \tag{3}$$

and

$$\begin{aligned} \sigma(R)^2 \quad = \quad & \Phi\left(-\frac{\tilde{\mu}(R)}{\tilde{\sigma}(R)}\right) - 2Exp\left(\tilde{\mu}(R) + \frac{\tilde{\sigma}(R)^2}{2}\right)\Phi\left(-\frac{\tilde{\mu}(R) + \tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right) \\ & + Exp\left(2\tilde{\mu}(R) + 2\tilde{\sigma}(R)^2\right)\Phi\left(-\frac{\tilde{\mu}(R) + 2\tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right) - \mu(R)^2, \end{aligned} \tag{4}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. Using observed values (consistent estimates from Moody's corporate bond default data) of $\mu(R)$ and $\sigma(R)^2$, we can solve for the unknowns $\tilde{\mu}(R)$ and $\tilde{\sigma}(R)$ in equations (3)-(4). Since there are two equations in two unknowns, we can solve for $\tilde{\mu}(R)$ and $\tilde{\sigma}(R)$ and hence identify the loss distribution for each rating category. Observe that the estimators of $(\mu(R), \sigma(R))$ are non-linear functions of $(\tilde{\mu}(R), \tilde{\sigma}(R))$. Thus, as long as (3)-(4) can be exactly solved for $(\mu(R), \sigma(R))$ using consistent estimates of $(\tilde{\mu}(R), \tilde{\sigma}(R))$, the solved values of $(\mu(R), \sigma(R))$ will be consistent by the Slutsky's theorem.

2.2 Asset Value Increase Needed for Enhancing Bond Rating

As discussed earlier, rating agencies implicitly specify a loss distribution for each rating category and assume that this distribution is unaffected by the rating accorded to a given bond issuer in that rating category. Given this assumption, ratings are not relative to a set of firms in an economy. Under this reasonable assumption, parameters of a distribution underlying each rating category, $(\tilde{\mu}(R), \tilde{\sigma}(R))$, are unaffected by a given issuer's credit enhancement policy. In this section, we determine consistent credit enhancement standards under this assumption.

An issuer can increase credit protection for its bondholders by reducing its existing debt or by equivalently increasing the investment in assets through new equity. Either of these ways will effectively reduce the risky debt relative to assets of the issuer. Let C_l^h denote the new equity infusion, per dollar of the current value of investment, that is necessary to increase the issuer's rating from l to h, where l or h is one of the six ratings (Aaa, Aa, A, Baa, Ba, B) for which we have the data available from MSR. After the infusion of new equity C_l^h, the issuer's current value of assets per dollar face value of existing debt will be $(1 + C_l^h)$. When these current assets are invested, they will generate a future value of assets,

$$(1 + C_l^h)V(l) \equiv \hat{V}(l). \tag{5}$$

If $C_l^h = 0$, the issuer will obviously stay l-rated, because its loss distribution will continue to correspond to that of an l-rated issuer. The bond loss rate remains the same whether C_l^h dollars of new equity capital is infused or, equivalently, the face value of bonds is raised from one dollar to $1/(1+C_l^h)$. This is because when the issuer infuses no new equity capital and increases the face value of bonds to $1/(1+C_l^h)$, the unconditional loss on all these bonds is $\max[0, 1/(1+C_l^h) - V(l)]$ and the bond loss rate (per dollar face value of bonds) is $\max[0, 1 - (1 + C_l^h)V(l)]$ which is the bond loss rate when new equity capital C_l^h is infused. Thus, credit quality (bond rating) can be consistently enhanced via new equity capital infusion or an equivalent debt reduction.

We now address whether or not there exists a positive amount of new equity C_l^h that a given l-rated issuer must raise to improve its bond rating, *consistently,* for instance, from $l = B$ to a higher rating $h = Aa$. A consistent enhancement in bond rating is defined by a reduction in the current expected bond loss rate of an issuer after it infuses new capital to that for higher rated bonds. Intuitively, C_l^h should be positive when h is a higher rating than l. A sufficient condition for C_l^h to be positive when rating l is desired to be improved to h is that $V(h)$ stochastically dominates $V(l)$ in the sense of second order stochastic dominance (SSD). While the SSD is a necessary condition, the first order stochastic dominance of $V(l)$ by $V(h)$ is sufficient for consistent credit enhancement standards. However, whether these conditions hold in the real world would depend on the consistency of bond ratings by the rating industry and on how default data are generated.

We now derive a formula for the unconditional expected loss rate, denoted $\hat{\mu}(l, C_l^h)$, on l-rated bonds after the issuer infuses new equity capital, C_l^h, into the entity:

$$\hat{\mu}(l, C_l^h) = E\left[\max\left\{0, 1 - (1 + C_l^h)V(l)\right\}\right]. \tag{6}$$

Using the formulas in the Appendix:

$$\begin{aligned}
\hat{\mu}(l, C_l^h) &= \int_0^{\frac{1}{1+C_l^h}} (1 - (1 + C_l^h)y)g(y)dy \\
&= \left[\Phi\left(\frac{-ln(1 + C_l^h) - \tilde{\mu}(l)}{\tilde{\sigma}(l)}\right) - (1 + C_l^h)Exp\left(\tilde{\mu}(l) + \frac{\tilde{\sigma}(l)^2}{2}\right) \times \right. \\
&\qquad \left. \Phi\left(\frac{-ln(1 + C_l^h) - (\tilde{\mu}(l) + \tilde{\sigma}(l)^2)}{\tilde{\sigma}(l)}\right)\right],
\end{aligned} \tag{7}$$

where $g(y)$ is the distribution function for $V(l)$.

To consistently determine C_l^h that raises the rating of an l-rated issuer's bonds to an h rating, we equate the reduced bond loss rate due to the new capital infusion with the loss rate for h-rated bonds:

$$\hat{\mu}(l, C_l^h) = \mu(h). \tag{8}$$

For example, the C_B^{Aa} that solves $\hat{\mu}(B, C_B^{Aa}) = \mu(Aa)$ is the new equity capital infusion needed by a B-rated issuer to reduce its unconditional expected bond loss rate to that of Aa-rated bonds. Using $R = h$ in (3) we obtain the right side of (8) and (7) gives the left side of (8) so that (7) can be solved for C_l^h. In the second step of our procedure, we solve (8) for C_l^h that consistently raises the issuer's bond rating from l to h. Observe that the distribution of $V(l)$ is not changed as a result of the new equity capital infusion. However, the distribution of the future asset value of an l-rated issuer, $\hat{V}(l)$, depends on the amount of new equity capital infused as well as $V(l)$, unlike in the standard industry procedure discussed earlier.

3 Estimation and Results

To obtain credit enhancement standards, we first estimate the asset value distribution parameters $(\tilde{\mu}(R), \tilde{\sigma}(R))$ using consistent estimates of the unconditional mean and variance of loss rates for bonds in each of the six rating categories reported in Moody's Special Report. We use "marginal" default rate and the severity of loss rate for bonds used in Moody's Special Report (1993) to obtain consistent estimates of the mean and variance of loss rates. The marginal default rate refers to the fraction of outstanding bond issuers that default (marginally drop out of the sample) in a given year in future reckoned from the date of issuance. For an illustration of our procedure, we use marginal rates of default for bonds with an age of three, four and five years. Credit enhancement standards are dependent on the age of bonds outstanding in an entity.

The marginal default rates in percent corresponding to Moody's sample of bond defaults over a 21-year period 1972-1992 are presented in Table 1. The same data without rounding are obtained from a companion paper (see Fons (1994)) from Moody's. For example, 2.67 percent of bonds rated Ba default three years after issuance. We observe from Table 1 that the rate of default is higher, the lower the rating. The aging does not seem to have an effect on marginal default rates for bonds issued 3 to 5 years ago.

To estimate the unconditional mean and variance of bond loss rates, observe that the probability of default, denoted $p(R)$, is the mean $E(Y)$ of a binomial random variable Y taking values $Y = 1$ for "default" and $Y = 0$ for "no default." The variance of this binomial random variable is equal to $Var(Y) = p(R)(1 - p(R))$. If x is the mean bond loss rate, conditional on a default, the unconditional random bond loss rate is xY. The unconditional mean bond loss rate is $E(xY) = xp(R)$, assuming that x is fixed. The variance of bond loss rate is $Var(xY) = x^2p(R)(1 - p(R))$.[7] Moody's special report (page 8) reports an estimate of x as .6, based on its proprietary corporate bond default database. Table 2 shows estimates of the unconditional mean and variance of loss rates $(\mu(R), \sigma(R))$ and the corresponding estimates of parameters of the asset value distribution $(\tilde{\mu}(R), \tilde{\sigma}(R))$ obtained by solving (3)-(4) for each rating and bond age.

We then solve for the amount of new equity capital to be raised by a bond issuer, C_l^h, to enhance bond rating from l to h. These results are presented in Table 3. For bonds with an age of 3 years, our estimates of the amount of new equity capital (as a fraction of current assets)

[7] If data on variance of x and on its covariance with Y were available, one could exploit these data to estimate $E(xY)$ and $Var(xY)$ in accordance with this data and then use the consistent estimates of unconditional mean and variance of bond loss rates in our two-step procedure for estimating credit enhancement standards.

to enhance the rating to *Aaa* are: .00418 for an *Aa*-rated, .00743 for an *A*-rated, .03022 for a *Baa*-rated, .18985 for a *Ba*-rated and .43259 for a *B*-rated issuer. The numbers are similar for other ages of bonds.

Table 4 shows the estimates under the standard procedure adopted by the rating industry. As discussed earlier, this procedure calculates the new equity capital infusion C_l^h by solving $(x - C_l^h)p(l) = x\,p(h)$, i.e., $C_l^h = x(1-p(h)/p(l))$. The results under this industry procedure presented in Table 4 show much larger amounts of new equity capital infusion than the corresponding numbers in Table 3.[8]

4 Conclusion

This paper presented a new procedure to consistently derive the amount of new equity capital that an issuer of bonds must raise to enhance the ratings of its bonds. This procedure exploits the same corporate bond default data as employed by the bond rating agencies like Moody's. We find that a relatively strong assumption made in the industry approach is perhaps responsible for obtaining very large amounts of new equity capital infusions to enhance the bond rating of an issuer. For example, the industry approach shows that an *Aa*-rated issuer must raise 40 cents for every dollar of its current assets to enhance its *Aa* rating to *Aaa*, as opposed to .4 cents per dollar of existing assets obtained by the new approach.

[8]Observe that the assumption that asset values are lognormally distributed may be strong and may not apply to actual data. Given sufficient data, however, the empirical distribution of asset values can be derived to conduct simulation. Since the author was unable to access proprietary default bond data from rating agencies, simulations could not be done.

The Appendix (The Moment of a Truncated Lognormal Variate)

For ready reference, we derive the conditional moment of a lognormally distributed random variable, y. Define $w \equiv ln(y)$ with a support $(0,\infty)$. Since y is lognormally distributed [suppressing the argument R],

$$w = ln(y) \sim N\left[\tilde{\mu}, \tilde{\sigma}\right]. \tag{A1}$$

It is useful to derive the expression for the general form of the conditional moment of y^n for $n = 0, 1, 2...$ as follows [letting $K = \frac{1}{\sqrt{2\pi}\tilde{\sigma}}$, and $g(y) = \frac{1}{\sqrt{2\pi}\tilde{\sigma}y} Exp\left(-\frac{1}{2\tilde{\sigma}^2}(ln(y)-\tilde{\mu})^2\right)$ the lognormal density] for $k > 0$]:

$$\begin{aligned}
\int_0^k y^n g(y)dy &= \int_{-\infty}^{ln(k)} K Exp(nw) Exp\left(-\frac{1}{2\tilde{\sigma}^2}(w-\tilde{\mu})^2\right) dw \\
&= \int_{-\infty}^{ln(k)} K Exp\left(-\frac{1}{2\tilde{\sigma}^2}\left[(w-\tilde{\mu})^2 - 2\tilde{\sigma}^2 nw\right]\right) dw \\
&= \int_{-\infty}^{ln(k)} K Exp\left(-\frac{1}{2\tilde{\sigma}^2}\left[w^2 + \tilde{\mu}^2 - 2w(\tilde{\mu}+n\tilde{\sigma}^2)\right]\right) dw \\
&= \int_{-\infty}^{ln(k)} K Exp\left(-\frac{1}{2\tilde{\sigma}^2}\left[\left(w-\left(\tilde{\mu}+n\tilde{\sigma}^2\right)\right)^2 + \tilde{\mu}^2 - \left(\tilde{\mu}+n\tilde{\sigma}^2\right)^2\right]\right) dw \\
&= \int_{-\infty}^{ln(k)} K Exp\left(-\frac{1}{2\tilde{\sigma}^2}\left[\left(w-\left(\tilde{\mu}+n\tilde{\sigma}^2\right)\right)^2 - 2n\tilde{\sigma}^2\left(\tilde{\mu}+\frac{n}{2}\tilde{\sigma}^2\right)\right]\right) dw \\
&= Exp\left(n\tilde{\mu} + \frac{1}{2}(n\tilde{\sigma})^2\right) \int_{-\infty}^{ln(k)} K Exp\left(-\frac{1}{2\tilde{\sigma}^2}\left[\left(w-\left(\tilde{\mu}+n\tilde{\sigma}^2\right)\right)^2\right]\right) dw \\
&= Exp\left(n\tilde{\mu} + \frac{1}{2}(n\tilde{\sigma})^2\right) \Phi\left(\frac{ln(k) - \left(\tilde{\mu}+n\tilde{\sigma}^2\right)}{\tilde{\sigma}}\right).
\end{aligned} \tag{A2}$$

Now,

$$\begin{aligned}
\mu(R) &= \int_0^1 (1-y)g(y)dy \\
&= \int_0^1 g(y)dy - \int_0^1 yg(y)dy \\
&= \Phi\left(-\frac{\tilde{\mu}(R)}{\tilde{\sigma}(R)}\right) - Exp\left(\tilde{\mu}(R) + \frac{\tilde{\sigma}(R)^2}{2}\right)\Phi\left(-\frac{\tilde{\mu}(R)+\tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right),
\end{aligned} \tag{A3}$$

Further,

$$\begin{aligned}
\sigma(R)^2 &= Var(\max(0, 1-V(R))) = E[(\max(0,1-V(R)))^2] - \mu(R)^2 \\
&= \int_0^1 (1-y)^2 g(y)dy - \mu(R)^2 \\
&= \int_0^1 g(y)dy - \int_0^1 2yg(y)dy + \int_0^1 y^2 g(y)dy - \mu(R)^2 \\
&= \Phi\left(-\frac{\tilde{\mu}(R)}{\tilde{\sigma}(R)}\right) - 2Exp\left(\tilde{\mu}(R) + \frac{\tilde{\sigma}(R)^2}{2}\right)\Phi\left(-\frac{\tilde{\mu}(R)+\tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right) \\
&\quad + Exp\left(2\tilde{\mu}(R) + 2\tilde{\sigma}(R)^2\right)\Phi\left(-\frac{\tilde{\mu}(R)+2\tilde{\sigma}(R)^2}{\tilde{\sigma}(R)}\right) - \mu(R)^2.
\end{aligned} \tag{A4}$$

Table 1
Marginal Corporate Bond Default Rates
Source: Moody's Special Report (1993)

Rating	Default Rates in percent for bonds aged		
	3 years	4 years	5 years
Aaa	0.04	0.09	0.10
Aa	0.12	0.12	0.11
A	0.18	0.16	0.21
Baa	0.56	0.52	0.50
Ba	2.67	2.69	2.19
B	5.52	4.79	4.89

Table 2
Estimates of Parameters

Rating	$\mu(R)$	$\sigma(R)$	$\tilde{\mu}(R)$	$\tilde{\sigma}(R)$
		Bonds aged 3 years		
Aaa	0.00024	0.00929	0.00047	0.00109
Aa	0.00072	0.01609	0.00138	0.00325
A	0.00108	0.01971	0.00207	0.00488
Baa	0.00336	0.03472	0.00557	0.01441
Ba	0.01602	0.07533	0.02174	0.06530
B	0.03312	0.10737	0.02949	0.12237
		Bonds aged 4 years		
Aaa	0.00054	0.01394	0.00103	0.00243
Aa	0.00072	0.01609	0.00138	0.00325
A	0.00096	0.01858	0.00180	0.00430
Baa	0.00312	0.03346	0.00535	0.01354
Ba	0.01614	0.07561	0.01892	0.06278
B	0.02874	0.10024	0.02822	0.10827
		Bonds aged 5 years		
Aaa	0.00060	0.01469	0.00112	0.00268
Aa	0.00066	0.01541	0.00123	0.00295
A	0.00126	0.02128	0.00231	0.00559
Baa	0.00300	0.03281	0.00522	0.01309
Ba	0.01314	0.06832	0.01639	0.05187
B	0.02934	0.10125	0.02581	0.10741

$\mu(R)$ and $\sigma(R)$ are, respectively, the unconditional mean and variance of bond loss rate for issuers rated $R = Aaa, Aa, A, Baa, Ba, B$. $\tilde{\mu}(R)$ and $\tilde{\sigma}(R)$ are, respectively, the mean and variance of the continuously compounded rate of return to assets backing bonds rated R.

Table 3
Estimates of New Equity for Credit Enhancement
Under the New Approach

To enhance to rating *Aaa* From Rating	New Equity as a fraction of Current Assets For Bonds Aged		
	3 years	4 years	5 years
Aa	0.00418	0.00205	0.00105
A	0.00743	0.00393	0.00587
Baa	0.03022	0.02273	0.02094
Ba	0.18985	0.16258	0.12530
B	0.43259	0.32760	0.32227

Table 4
Estimates of New Equity for Credit Enhancement
Under the Standard Industry Approach

To enhance to rating *Aaa* From Rating	New Equity as a fraction of Current Assets For Bonds Aged		
	3 years	4 years	5 years
Aa	0.4000	0.1500	0.0545
A	0.4667	0.2625	0.3143
Baa	0.5571	0.4962	0.4800
Ba	0.5910	0.5799	0.5726
B	0.5957	0.5887	0.5877

References

Fons, Jerome S., 1994, “Using Default Rates to Model Term Structure of Credit Risk,” **Moody’s Investors Service**, New York, NY 10007.

Moody’s Investors Service, 1991, “Rating Cash Flow Transactions Backed by Corporate Debt–Appendix A,” **Structured Finance Research and Company**, New York, March.

Moody’s Investors Service, 1993, “Moody’s Approach to Evaluating Derivative Products Subsidiaries,” **Moody’s Special Comment**, New York, October.

Moody’s Investors Service Structured Finance Research and Company, 1991, “Rating Cash Flow Transactions Backed By Corporate Debt,” **Appendix A: Mathematics of Calculating Portfolio Credit Risk**, March, New York, NY 10007.

Moody’s Investors Service, 1993, “Moody’s Special Report: Corporate Bond Defaults and Default Rates 1970-1992,” January, New York, NY 10007.

Lucas, R.E., 1976, “Econometric Policy Evaluation: A Critique,” in K. Brunner and A.H.Meltzer (eds.), **The Phillips Curve and Labor Markets**, Carnegie-Rochester Conference Series on Public Policy, vol. 1, North-Holland, Amsterdam.